

VEKTORANALYS

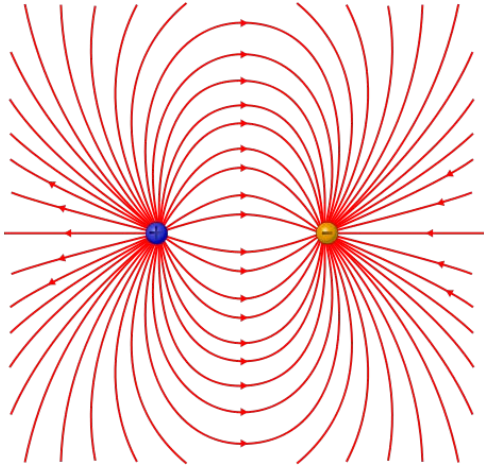
Kursvecka 3

GAUSS'S THEOREM and STOKES'S THEOREM

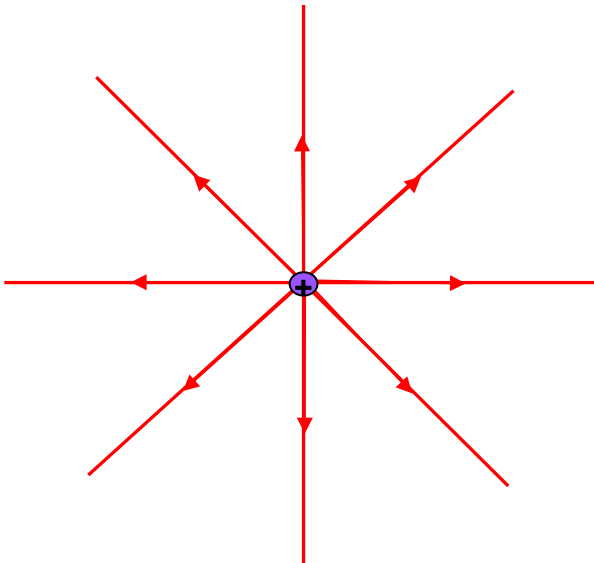
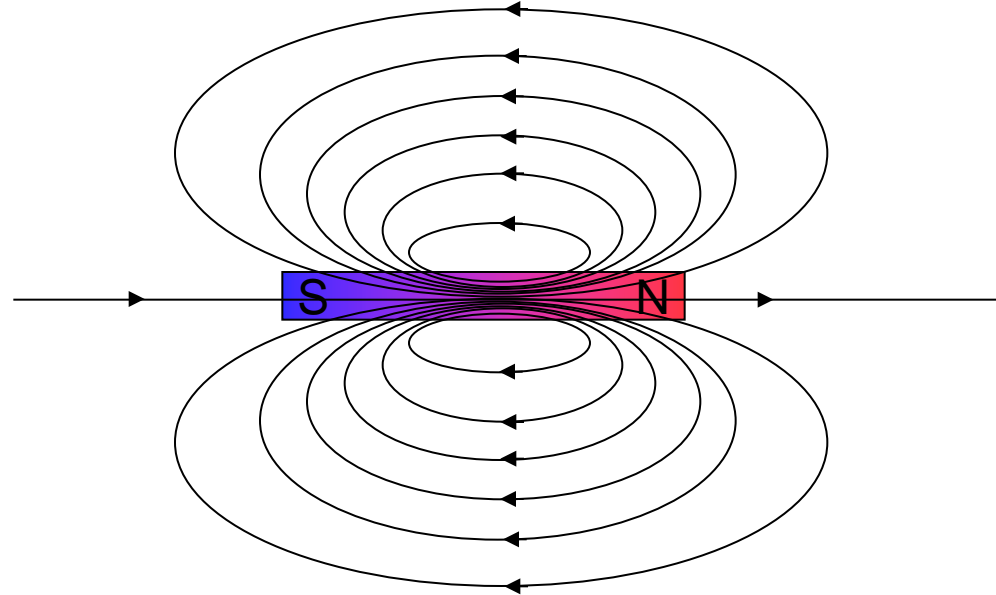
Kapitel 6-7
Sidor 51-82

TARGET PROBLEM

ELECTRIC FIELD



MAGNETIC FIELD



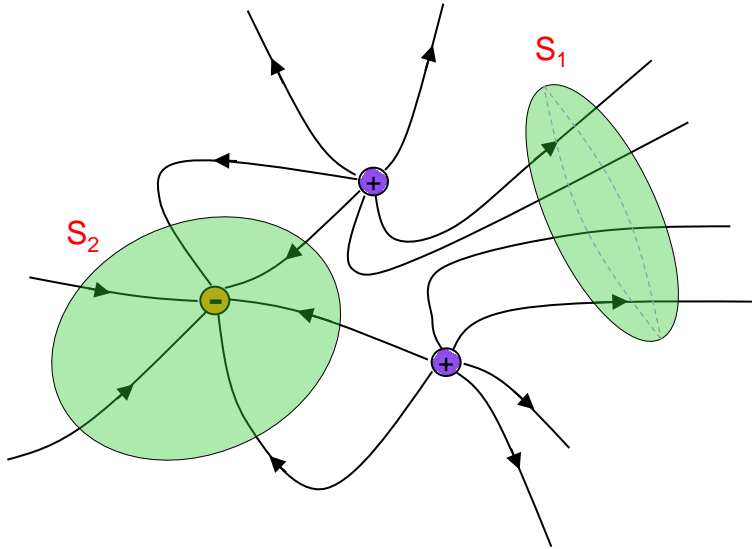
**Magnetic monopoles
do not exist in nature.**

- How can we express this information for \vec{E} and \vec{B} using the mathematical formalism?

TARGET PROBLEM

Let's consider some ELECTRIC CHARGES

and two closed surfaces, S_1 and S_2



S_1 does not contain any charge.
It has no sources and no sinks:
no field lines destroyed and
no field lines created inside S_1

$$\iint_{S_1} \vec{E} \cdot d\vec{S} = 0$$

S_2 do contain a charge.
It has a sink. Field lines are destroyed inside S_2

$$\iint_{S_2} \vec{E} \cdot d\vec{S} \neq 0$$

$$\int_S \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0} \quad \text{Gauss's law}$$

(see the 6th week of this course for details or "Teoretisk elektroteknik")

We want to find the differential form of the Gauss's law.

(i.e. to express the Gauss's law without using integrals)

• to introduce the divergence of a vector field \vec{A} , $\text{div} \vec{A}$

• the Gauss's theorem $\iint_S \vec{A} \cdot d\vec{S} = \iiint_V \text{div} \vec{A} dV$

THE DIVERGENCE (DIVERGENSEN)

In cartesian coordinates, the divergence of a vector field \bar{A} is:

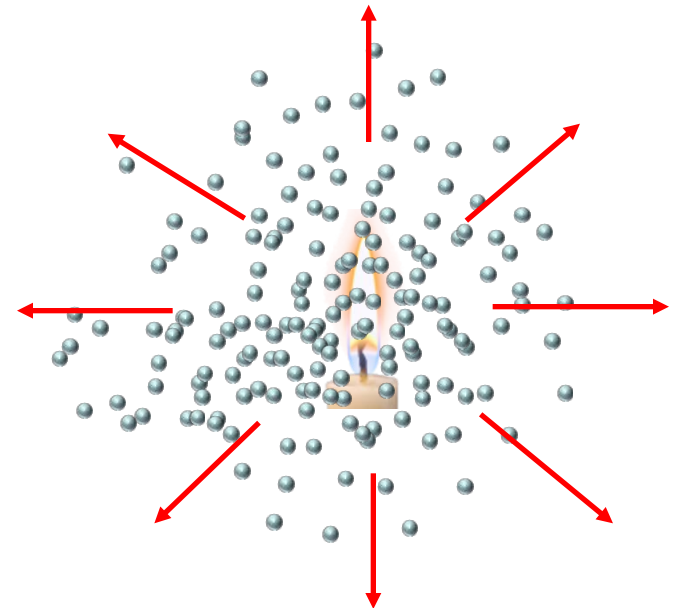
DEFINITION

$$\operatorname{div}\bar{A} \equiv \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (1)$$

It is a measure of **how much the field diverges (or converges) from (to) a point.**

EXAMPLE:

- Assume that \bar{A} is the velocity field of a gas.
- If heated, the gas will expand creating a velocity field that will diverge.
- The divergence of \bar{A} in the heating point will be positive
- If cooled, the gas will contract creating a velocity field that will converge on the cooling position. The divergence will be negative
- The heating position is a **source** of the velocity field and the cooling position is a **sink** of the velocity



The divergence is a measurement of sources or sinks
(this will be more clear using the Gauss's theorem)

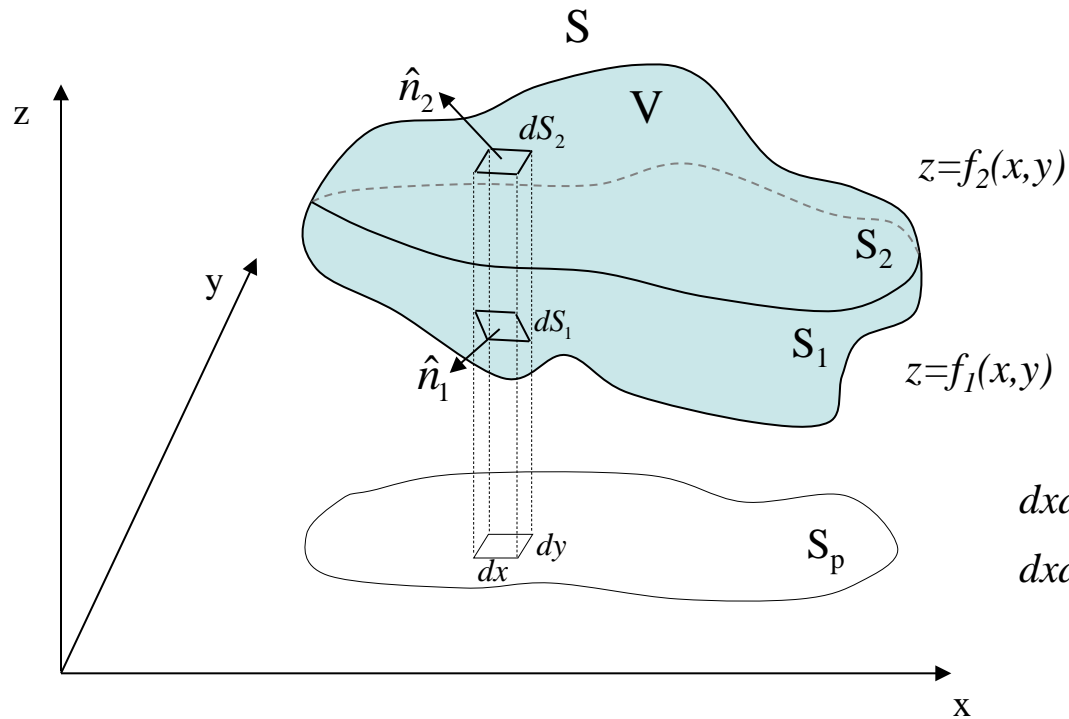
THE GAUSS'S THEOREM

$$\boxed{\iint_S \bar{A} \cdot d\bar{S} = \iiint_V \text{div} \bar{A} dV}$$



(2)

where **S is a closed surface** that forms the boundary of the volume V and \bar{A} is a continuously differentiable vector field defined on V.



$$dxdy = dS_2 \hat{n}_2 \cdot \hat{e}_z = d\bar{S}_2 \cdot \hat{e}_z$$

$$dxdy = -dS_1 \hat{n}_1 \cdot \hat{e}_z = -d\bar{S}_1 \cdot \hat{e}_z$$

THE GAUSS'S THEOREM

PROOF

$$\begin{aligned} \iiint_V \operatorname{div} \bar{A} dV &= \iiint_V \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dx dy dz = \\ &= \iiint_V \frac{\partial A_x}{\partial x} dx dy dz + \iiint_V \frac{\partial A_y}{\partial y} dx dy dz + \iiint_V \frac{\partial A_z}{\partial z} dx dy dz \end{aligned}$$

Let's calculate the last term:

$$\iiint_V \frac{\partial A_z}{\partial z} dx dy dz = \iint_{S_p} dx dy \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial A_z}{\partial z} dz = \iint_{S_p} [A_z(x, y, f_2(x, y)) - A_z(x, y, f_1(x, y))] dx dy =$$

$dx dy$ is the projection on S_p of the small element surfaces on dS_1 and dS_2 .

Therefore: $dx dy = -\hat{e}_z \cdot \hat{n}_1 dS_1 = \hat{e}_z \cdot \hat{n}_2 dS_2$

$$= \iint_{S_2} A_z(x, y, f_2(x, y)) \hat{e}_z \cdot \hat{n}_2 dS_2 + \iint_{S_1} A_z(x, y, f_1(x, y)) \hat{e}_z \cdot \hat{n}_1 dS_1 = \iint_S A_z \hat{e}_z \cdot \hat{n} dS$$

Which means:

$$\iiint_V \frac{\partial A_z}{\partial z} dV = \iint_S A_z \hat{e}_z \cdot \hat{n} dS \quad (3)$$

THE GAUSS'S THEOREM

PROOF

In the same way we get:

$$\iiint_V \frac{\partial A_x}{\partial x} dV = \iint_S A_x \hat{e}_x \cdot \hat{n} dS \quad (4)$$

$$\iiint_V \frac{\partial A_y}{\partial y} dV = \iint_S A_y \hat{e}_y \cdot \hat{n} dS \quad (5)$$

Adding together equations (3), (4) and (5) we finally obtain:

$$\begin{aligned} \iiint_V \operatorname{div} \bar{A} dV &= \iiint_V \frac{\partial A_x}{\partial x} dx dy dz + \iiint_V \frac{\partial A_y}{\partial y} dx dy dz + \iiint_V \frac{\partial A_z}{\partial z} dx dy dz = \\ &= \iint_S A_x \hat{e}_x \cdot \hat{n} dS + \iint_S A_y \hat{e}_y \cdot \hat{n} dS + \iint_S A_z \hat{e}_z \cdot \hat{n} dS = \iint_S \bar{A} \cdot d\bar{S} \end{aligned}$$

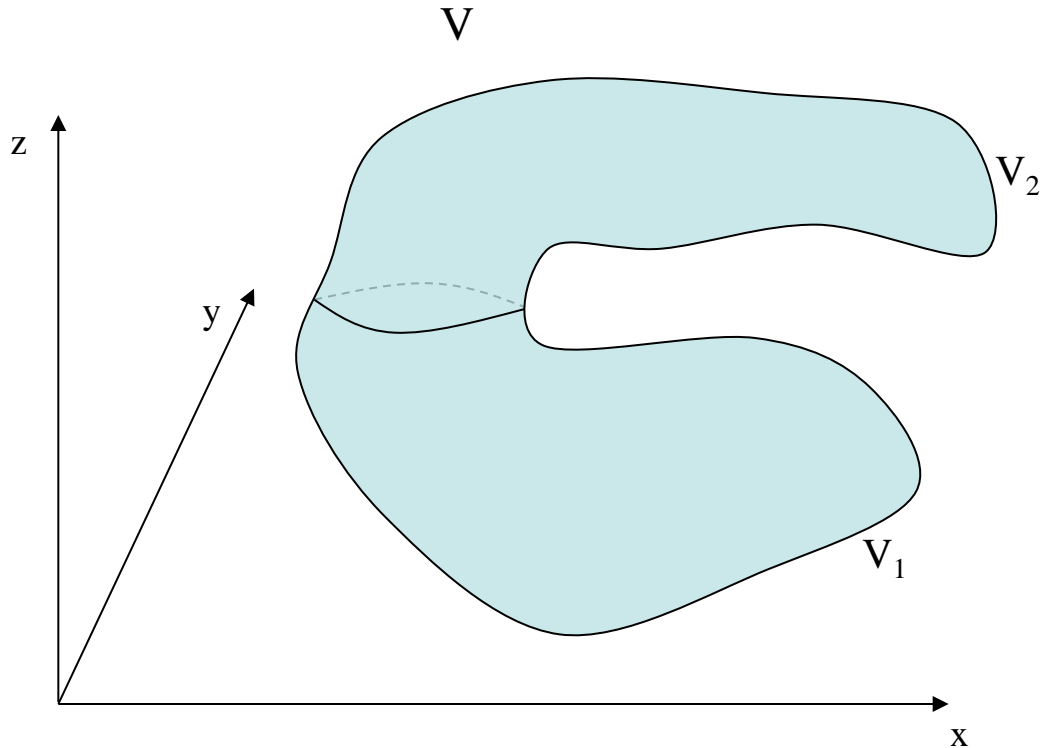
Rearrange in logic order the steps to prove the Gauss's theorem

- Add all the three terms together in order to obtain the flux of \bar{A} .
- Write down the volume integral of $div\bar{A}$
- Consider the projection of the surface element on the xy plane, it will be $dxdy$. The projection will identify a infinitesimal surface element (dS_2) on the lower surface.
 - Re-arrange the integrals in dS_1 and dS_2 in order to have obtain a flux integral of $(0,0,A_z)$.
- Consider a closed surface.
- Split the volume integral into three terms.
 - Consider only the term which depends on the z-derivative of A_z .
- Repeat the same for the terms which depend on the x-derivative of A_x and on the y-derivative of A_y .
 - Express $dxdy$ in order to obtain dS_1 and dS_2 .
 - Remove the z-derivative by solving the integral in dz .
What will remain is just the integral in $dxdy$.
- Divide the surface in two parts, an upper surface and a lower surface and consider an infinitesimal surface element dS_1 on the upper surface.
- Write the expression that relates $dxdy$ to dS_1 and dS_2 .

THE GAUSS'S THEOREM

PROOF

What if we consider a more complicated volume?



We divide the volume V in smaller and “simpler” volumes

$$V = V_1 + V_2 + \dots = \sum_i V_i$$

$$\iiint_V \operatorname{div} A dV = \sum_i \iiint_{V_i} \operatorname{div} A dV =$$

$$\sum_i \iint_{S_i} A \cdot dS = \iint_S A \cdot dS$$

PHYSICAL INTERPRETATION

Suppose that $\vec{v}(\vec{r})$ is the velocity field of a gas

Let's apply the Gauss' theorem to a volume V of the gas

$$\oiint_S \vec{v} \cdot d\vec{S} = \iiint_V \operatorname{div}(\vec{v}) dV$$

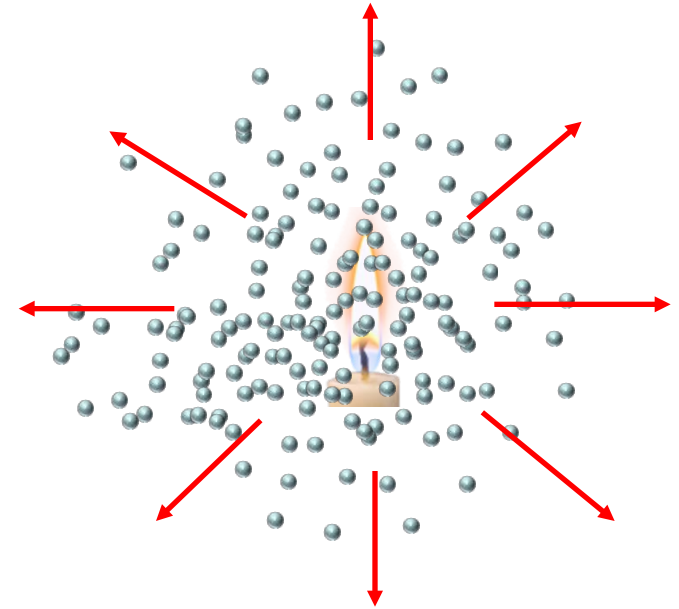
This term is the volume per second [m^3/s]
that flows out (*in*) from the closed surface S

If there are no sinks and no sources,
then no gas flows in S
and no gas flows out from S .
This implies that the flow $\oiint_S \vec{v} \cdot d\vec{S}$ is zero.
Therefore, $\operatorname{div}(\vec{v}) = 0$

$\operatorname{div}(\vec{v}) = 0 \Rightarrow$ No sink and no source

$\operatorname{div}(\vec{v}) < 0 \Rightarrow$ flux is destroyed
and there is a sink

$\operatorname{div}(\vec{v}) > 0 \Rightarrow$ flux is created
and there is a source



TARGET PROBLEM

**Magnetic monopoles do not exist in nature.
What this implies, in terms of the magnetic field?**

Magnetic monopoles do not exist \Rightarrow the flux of **B** is zero

Let's apply the Gauss's theorem to the magnetic field:



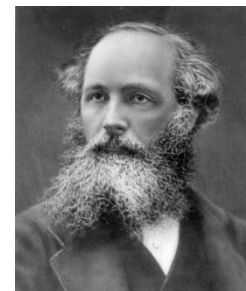
Gauss \Rightarrow
$$\left. \begin{aligned} \iint_S \vec{B} \cdot d\vec{S} &= \iiint_V \operatorname{div} \vec{B} dV \\ \iint_S \vec{B} \cdot d\vec{S} &= 0 \end{aligned} \right\} \Rightarrow \operatorname{div} \vec{B} = 0$$

\uparrow

Exercise: apply the Gauss's theorem

to the Gauss's law:
$$\iint \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$$

*One of the four
Maxwell's
equations*



WHICH STATEMENT IS WRONG?

- 1- The divergence of a vector field is a scalar (yellow)
- 2- The divergence is related to a measurement of the flux (red)
- 3- The Gauss' theorem translates a surface integral into a volume integral (green)
- 4- The Gauss' theorem can be applied also to a non closed surface (blue)

VEKTORANALYS

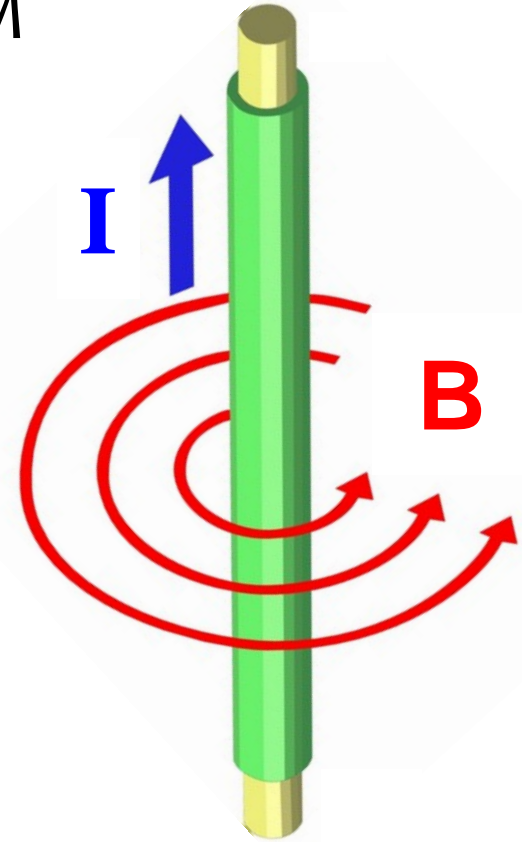
CURL (ROTATIONEN)

and

STOKES'S THEOREM

TARGET PROBLEM

- The current \bar{I} is flowing in a conductor
- How to calculate the magnetic field?



We need:

- Definition of the “curl” (or rotor) of a vector field

$$\text{rot} \bar{A}$$

- The **Stokes' theorem**

$$\oint_L \bar{A} \cdot d\mathbf{r} = \iint_S \text{rot} \bar{A} \cdot d\bar{S}$$

- A law that relates the current with the magnetic field:

the fourth Maxwell's equation (with static electric field): $\text{rot} \bar{B} = \mu_0 \bar{j}$

(see “Teoretisk elektroteknik”)

THE CURL (ROTATIONEN) $\text{rot } \bar{A}$

DEFINITION (in cartesian coordinate)

$$\text{rot } \bar{A} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

rot stands for “rotation”

In fact, the curl is a measure of how much the direction of a vector field changes in space, i.e. how much the field “rotates”.

In every point of the space, $\text{rot } \bar{A}$ is a vector whose length and direction characterize the rotation of the field \bar{A} .

The direction is the axis of rotation of \bar{A}

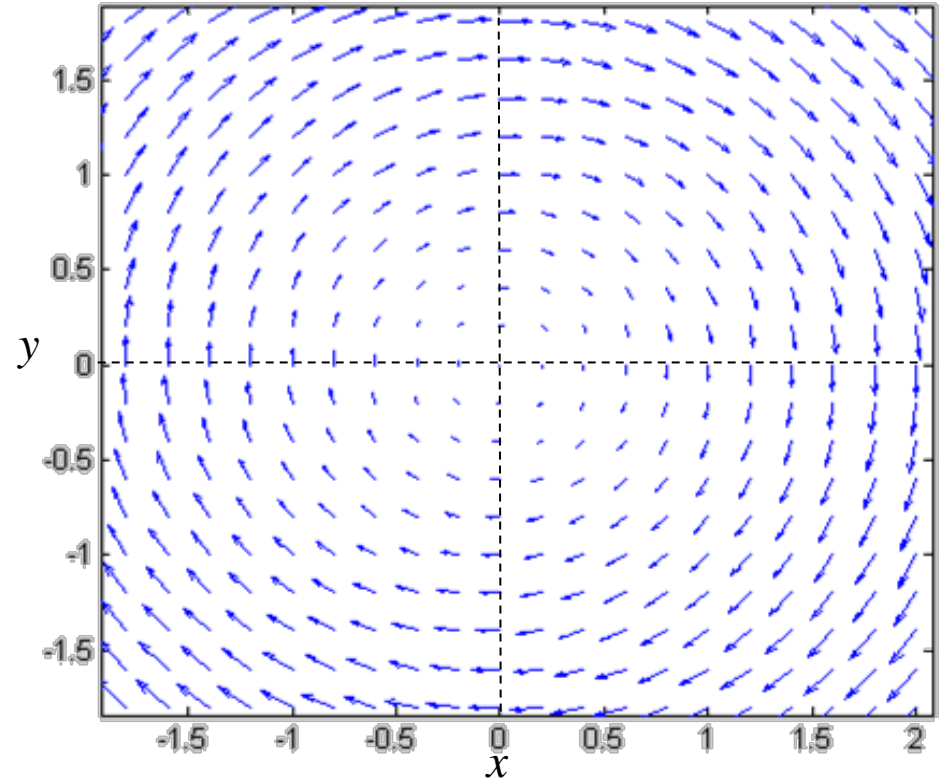
The magnitude is the magnitude of rotation of \bar{A}

THE CURL $\text{rot}\bar{A}$

EXAMPLE

$$\bar{A}(x, y, z) = (y, -x, 0)$$

Exercise: calculate the curl of \bar{A}



Direction: the direction is the axis of rotation, i.e. perpendicular to the plane of the figure

The sign (negative, in this case) is determined by the right-hand rule

Magnitude: the amount of rotation

In this example, it is constant and independent of the position, i.e. the amount of rotation is the same at any point.

THE CURL $\overline{rot\bar{A}}$

PHYSICAL INTERPRETATION

Consider the rotation of a rigid body around the z -axis

The coordinates of a point **P** on the body located at the distance a from the z -axis and at $z=z_0$ changes in time:

$$x(t) = a \cos \omega t$$

$$y(t) = a \sin \omega t$$

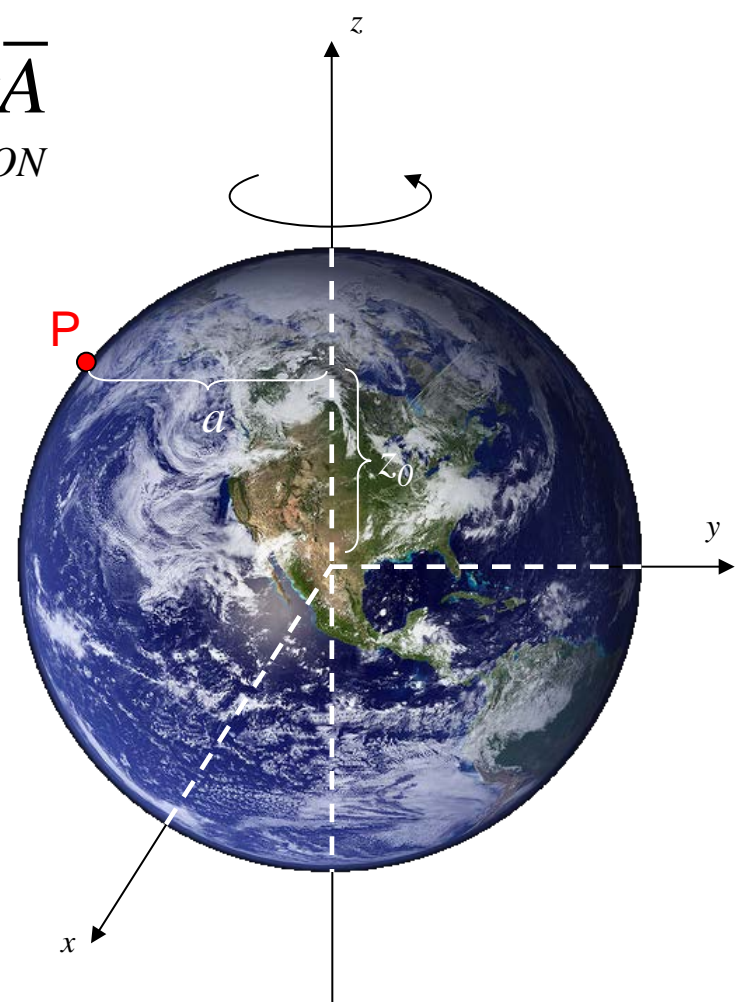
$$z = z_0$$

The velocity of the point **P** is:

$$\left. \begin{aligned} v_x(t) &= -a\omega \sin \omega t = -\omega y(t) \\ v_y(t) &= a\omega \cos \omega t = \omega x(t) \\ v_z &= 0 \end{aligned} \right\} \Rightarrow \bar{v} = (-\omega y, \omega x, 0)$$

Therefore $\overline{rot\bar{v}} = (0, 0, 2\omega)$

$$\bar{\omega} = \frac{1}{2} \overline{rot\bar{v}}$$

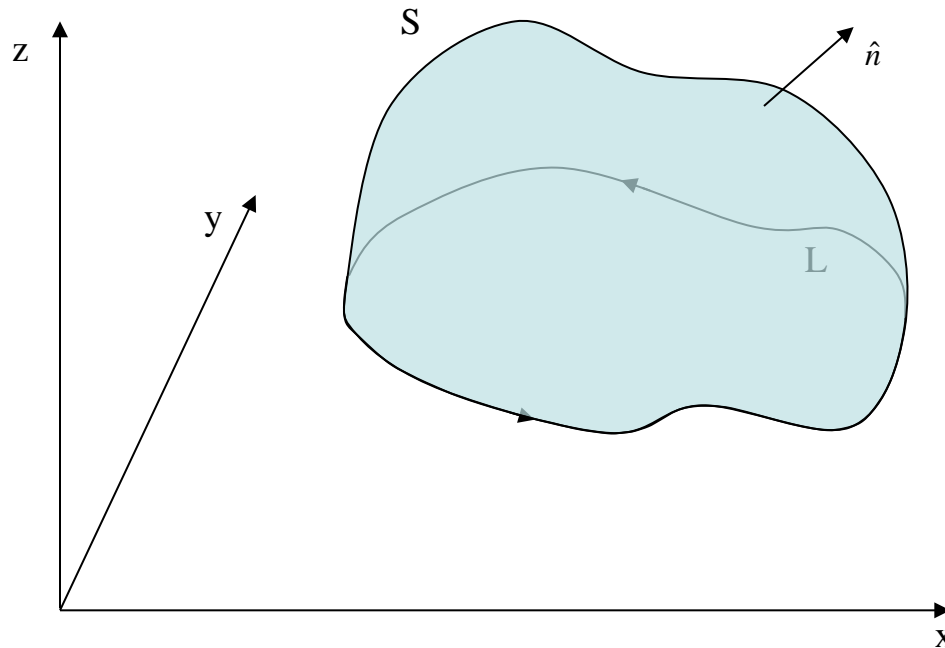


THE STOKES' THEOREM

$$\oint_L \bar{A} \cdot d\bar{r} = \iint_S \text{rot} \bar{A} \cdot d\bar{S}$$



where \bar{A} is a vector field, **L is a closed curve** and S is a surface whose boundary is defined by L .
 \bar{A} must be continuously differentiable on S



THE STOKES' THEOREM

PROOF

Five steps:

1. We divide S in “many” “smaller”
(infinitesimal) surfaces:

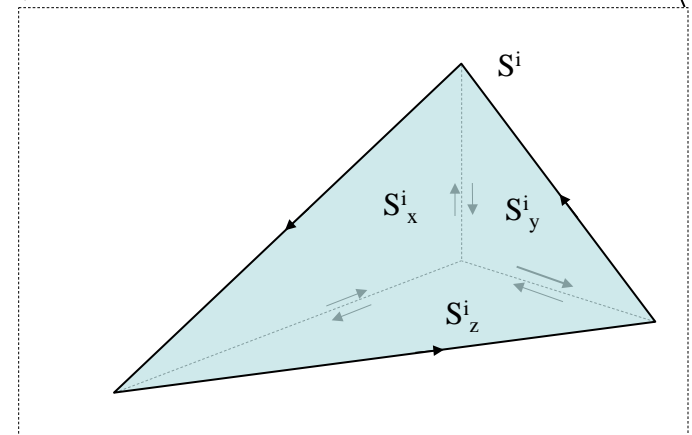
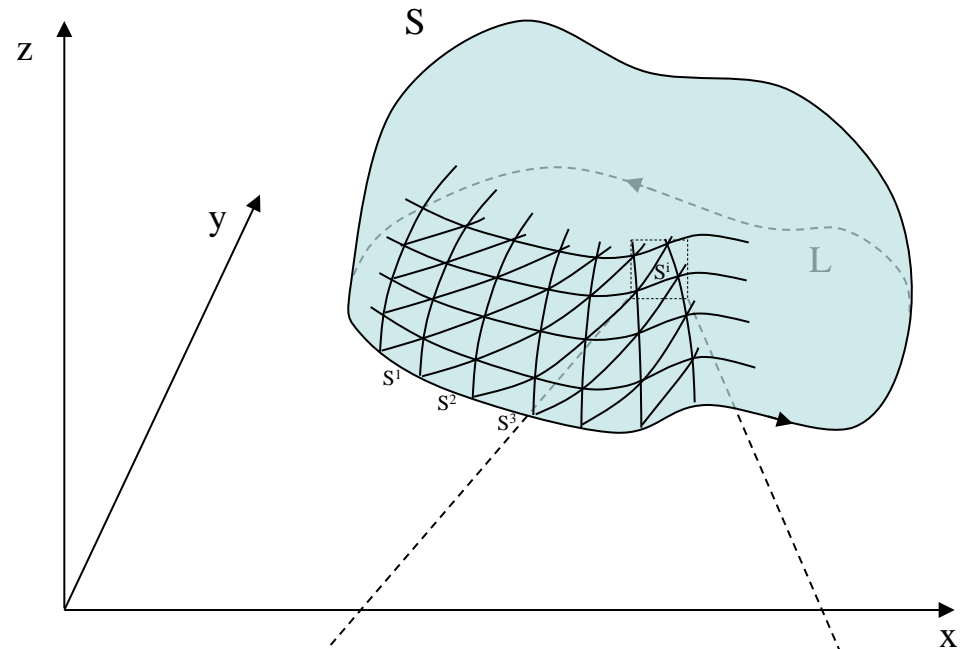
$$S = \sum_i S^i$$

2. We project S^i on:
 - the xy-plane S^i_z
 - the yz-plane S^i_x
 - the xz-plane S^i_y

3. We prove the Stokes' theorem on S^i_z ,
(the only difficult part)

4. We add the results for the projections together
and we obtain the Stokes' theorem on S^i

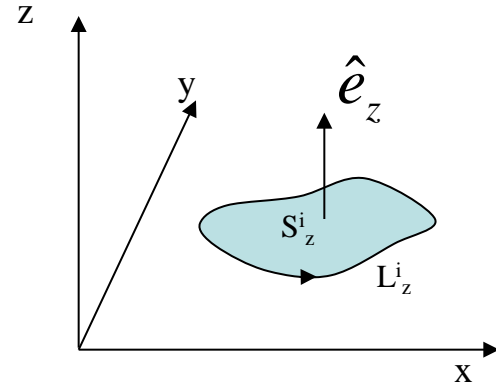
5. We add the results for S^i together
and we obtain the Stokes' theorem on S



THE STOKES' THEOREM

PROOF

Let's consider the plane surface S_z^i located in the xy-plane (i.e. $z=\text{constant}=z_0$) with boundary defined by the curve L_z^i



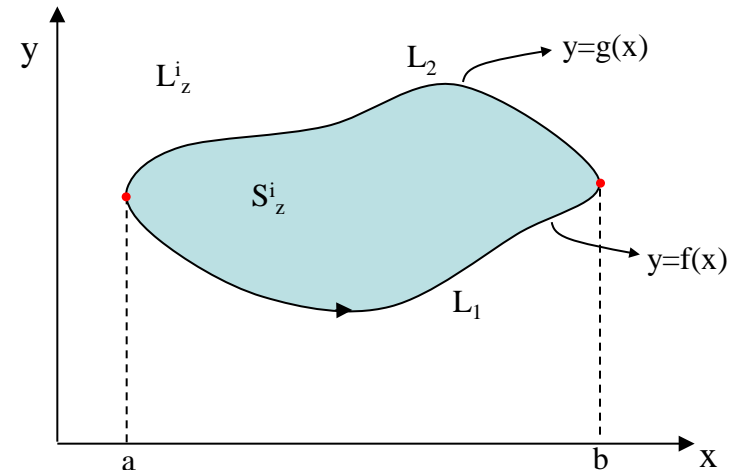
Let's calculate $\oint_{L_z^i} \bar{A} \cdot d\bar{r}$

$$\oint_{L_z^i} \bar{A} \cdot d\bar{r} = \underbrace{\oint_{L_z^i} A_x(x, y, z_0) dx}_{\text{Term 1}} + \underbrace{\oint_{L_z^i} A_y(x, y, z_0) dy}_{\text{Term 2}} + \underbrace{\oint_{L_z^i} A_z(x, y, z_0) dz}_{\text{Term 3}}$$

Term 3 = 0 ($z=\text{constant!} \Rightarrow dz=0$)

Term 1

$$\begin{aligned} \oint_{L_z^i} A_x(x, y, z_0) dx &= \oint_{L_1+L_2} A_x(x, y, z_0) dx = \\ &= \int_{L_1} A_x(x, y, z_0) dx + \int_{L_2} A_x(x, y, z_0) dx = \\ &= \int_a^b A_x(x, f(x), z_0) dx + \int_b^a A_x(x, g(x), z_0) dx = \end{aligned}$$



THE STOKES' THEOREM

PROOF

$$= \int_a^b A_x(x, f(x), z_0) dx - \int_a^b A_x(x, g(x), z_0) dx = \int_a^b [A_x(x, f(x), z_0) - A_x(x, g(x), z_0)] dx =$$

$$\int_a^b \int_{g(x)}^{f(x)} \frac{\partial A_x(x, y, z_0)}{\partial y} dx dy = - \int_a^b \int_{f(x)}^{g(x)} \frac{\partial A_x}{\partial y} dx dy = - \iint_{S_z^i} \frac{\partial A_x}{\partial y} dx dy$$

Therefore we get:

Term 1 $\oint_{L_z^i} A_x(x, y, z_0) dx = - \iint_{S_z^i} \frac{\partial A_x}{\partial y} dx dy$

In a similar way:

Term 2 $\oint_{L_z^i} A_y(x, y, z_0) dx = \iint_{S_z^i} \frac{\partial A_y}{\partial x} dx dy$

It is the z-component of $rot \bar{A}$!!

Adding **Term 1**, **Term 2** and **Term 3**:

$$\oint_{L_z^i} \bar{A} \cdot d\bar{r} = \iint_{S_z^i} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dx dy$$

THE STOKES' THEOREM

So can rewrite it as:

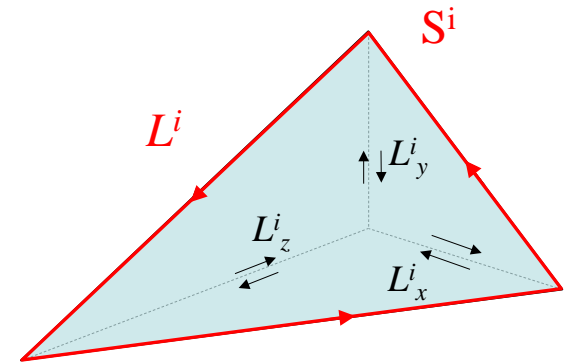
$$\oint_{L_z^i} \bar{A} \cdot d\bar{r} = \iint_{S_z^i} (\text{rot}\bar{A})_z dx dy = \iint_{S^i} (\text{rot}\bar{A})_z \hat{e}_z \cdot d\bar{S}$$

$$\underbrace{dx dy = \hat{e}_z \cdot \hat{n} dS = \hat{e}_z \cdot d\bar{S}}$$

In a similar way we have:

$$\oint_{L_y^i} \bar{A} \cdot d\bar{r} = \iint_{S^i} (\text{rot}\bar{A})_y \hat{e}_y \cdot d\bar{S}$$

$$\oint_{L_x^i} \bar{A} \cdot d\bar{r} = \iint_{S^i} (\text{rot}\bar{A})_x \hat{e}_x \cdot d\bar{S}$$



Now let's add everything together:

$$\oint_{L_x^i} \bar{A} \cdot d\bar{r} + \oint_{L_y^i} \bar{A} \cdot d\bar{r} + \oint_{L_z^i} \bar{A} \cdot d\bar{r} = \oint_{L^i} \bar{A} \cdot d\bar{r}$$

$$\iint_{S^i} (\text{rot}\bar{A})_x \hat{e}_x \cdot d\bar{S} + \iint_{S^i} (\text{rot}\bar{A})_y \hat{e}_y \cdot d\bar{S} + \iint_{S^i} (\text{rot}\bar{A})_z \hat{e}_z \cdot d\bar{S} = \iint_{S^i} \text{rot}\bar{A} \cdot d\bar{S}$$

Rearrange in logic order the steps to prove the Stokes' theorem

- Consider only the integral in dx and prove that $\int_{L_z^i} A_x(x, y, z_0) dx = - \iint_{S_z^i} \frac{\partial A_x}{\partial y} dx dy$
- add together the expressions for the integrals in S_x^i to S_y^i and S_z^i obtaining: $\int_{L^i} \bar{A} \cdot d\bar{r} = \iint_{S^i} \text{rot} \bar{A} \cdot d\bar{S}$
- Prove the Stokes' theorem on S_z^i :
 - Write the line integral of the vector field along the boundary of S_z^i and split the integral into three terms.
- Consider a closed path and a surface whose boundary is defined by the closed path.
- Divide the surface in small areas S^i and consider the projection of S^i on the xy, yz, xz planes
 - Repeat the same for the integral in dy and dz
- Prove the Stokes' theorem on S : add together all the expressions obtained for S^i
 - Rewrite $dx dy$ to obtain $\int_{L_z^i} \bar{A} \cdot d\bar{r} = \iint_{S^i} (\text{rot} \bar{A})_z \hat{e}_z \cdot d\bar{S}$
- Prove the Stokes' theorem on S^i :
 - Add the three integrals in dx, dy and dz to obtain $\int_{L_z^i} \bar{A} \cdot d\bar{r} = \iint_{S_z^i} (\text{rot} \bar{A})_z dx dy$
 - Repeat the same procedure for S_x^i and S_y^i

THE STOKES' THEOREM

PROOF

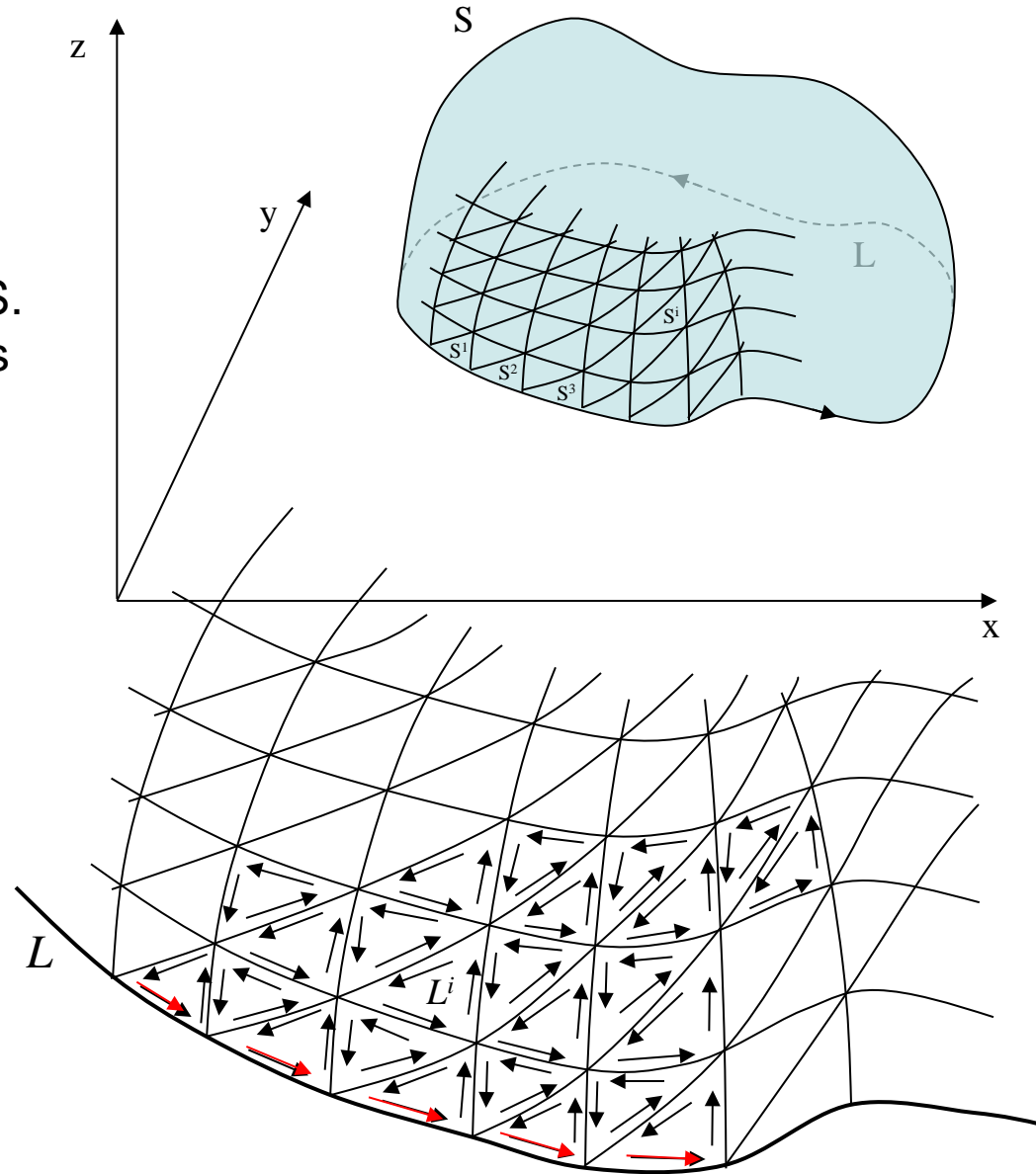
$$\oint_{L^i} \bar{A} \cdot d\bar{r} = \iint_{S^i} \text{rot} \bar{A} \cdot d\bar{S}$$

But we are interested in the whole S.
So we add these small contributions
altogether:

$$\sum_i \iint_{S^i} \text{rot} \bar{A} \cdot d\bar{S} = \iint_S \text{rot} \bar{A} \cdot d\bar{S}$$

$$\sum_i \int_{L^i} \bar{A} \cdot d\bar{r} = \int_L \bar{A} \cdot d\bar{r}$$

$$\oint_L \bar{A} \cdot d\bar{r} = \iint_S \text{rot} \bar{A} \cdot d\bar{S}$$

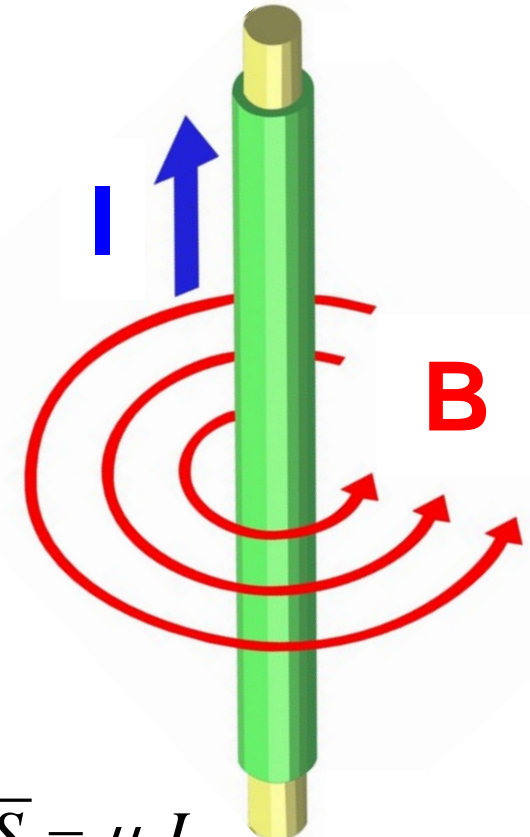


TARGET PROBLEM

Now we can calculate the magnetic field \vec{B} at a distance a from the conductor.

Ampere's law $\text{rot}\vec{B} = \mu_0 \vec{j}$

Where \vec{j} is the current density:



$$\oint_L \vec{B} \cdot d\vec{r} = \iint_S \text{rot}\vec{B} \cdot d\vec{S} = \iint_S \mu_0 \vec{j} \cdot d\vec{S} = \mu_0 \iint_S \vec{j} \cdot d\vec{S} = \mu_0 I$$

Stokes



Ampere



$$I = \iint_S \vec{j} \cdot d\vec{S}$$

THE GREEN FORMULA IN THE PLANE

THEOREM (7.1 in the textbook)

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_L (P dx + Q dy)$$

PROOF

We can start from Stokes' theorem $\oint_L \bar{A} \cdot d\bar{r} = \iint_S \text{rot } \bar{A} \cdot d\bar{S}$

$$\left. \begin{aligned} \oint_L \bar{A} \cdot d\bar{r} &= \oint (A_x dx + A_y dy + A_z dz) = \oint (A_x dx + A_y dy) \\ &\quad \uparrow \\ &\quad \text{But we are in a plane,} \\ &\quad \text{so we can assume } A=(A_x, A_y, 0) \end{aligned} \right\} \iint_D \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dx dy = \oint_L (A_x dx + A_y dy)$$

$$\iint_S \text{rot } \bar{A} \cdot d\bar{S} = \iint_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \underbrace{\hat{e}_z \cdot \hat{e}_z}_{=1} dx dy$$

$$\begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & 0 \end{vmatrix}$$

which is the Green formula
for $P=A_x$ and $Q=A_y$

CURL FREE FIELD AND SCALAR POTENTIAL

(*virvelfria fält och skalär potential*)

DEFINITION: A vector field \bar{A} is “curl free” if $\text{rot } \bar{A} = 0$ *Sometimes called “irrotational”*

THEOREM (7.5 in the textbook)

$\text{rot } \bar{A} = 0 \Leftrightarrow$ has a scalar potential ϕ , $\bar{A} = \text{grad } \phi$

PROOF

$$(1) \text{ rot } \bar{A} = 0$$

$$\oint_L \bar{A} \cdot d\bar{r} = \iint_S \text{rot } \bar{A} \cdot d\bar{S} = 0$$

If the circulation is zero, then the field is conservative and has a scalar potential. *See theorem 4.5 in the textbook.*

$$(2) \bar{A} = \text{grad } \phi$$

$$\text{rot } \bar{A} = \text{rot } \text{grad } \phi = \text{rot} \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \left(\frac{\partial}{\partial y} \frac{\partial \phi}{\partial z} - \frac{\partial}{\partial z} \frac{\partial \phi}{\partial y}, \dots, \dots \right) = (0, 0, 0)$$

SOLENOIDAL FIELD AND VECTOR POTENTIAL

DEFINITION: A vector field \bar{B} is called **solenoidal** if $div\bar{B} = 0$

DEFINITION: The vector field \bar{B} has a vector potential \bar{A} if , $\bar{B} = rot\bar{A}$

THEOREM (7.7 in the textbook) \bar{B} has a vector potential \bar{A} , $\bar{B} = rot\bar{A} \Leftrightarrow div\bar{B} = 0$

PROOF

$$(1) \bar{B} \text{ has a vector potential} \Rightarrow \bar{B} = rot\bar{A} \Rightarrow div\bar{B} = div(rot\bar{A}) = 0$$

$$(2) div\bar{B} = 0$$

Let's try to find a solution \bar{A} to the equation $\bar{B} = rot\bar{A}$

We start looking for a particular solution A^* of this kind:

$$\bar{A}^* = (A_x^*(x, y, z), A_y^*(x, y, z), 0)$$

CURL FREE FIELD AND SCALAR POTENTIAL

PROOF

Assuming $\bar{B} = \text{rot} \bar{A}$ we obtain:

$$-\frac{\partial A_y^*}{\partial z} = B_x \quad \Rightarrow \quad A_y^*(x, y, z) = -\int_{z_0}^z B_x(x, y, z) dz + F(x, y)$$

$$\frac{\partial A_x^*}{\partial z} = B_y \quad \Rightarrow \quad A_x^*(x, y, z) = \int_{z_0}^z B_y(x, y, z) dz + G(x, y)$$

$$\frac{\partial A_y^*}{\partial x} - \frac{\partial A_x^*}{\partial y} = B_z \quad \Rightarrow \quad -\int_{z_0}^z \frac{\partial B_x}{\partial x} dz + \frac{\partial F}{\partial x} - \int_{z_0}^z \frac{\partial B_y}{\partial y} dz - \frac{\partial G}{\partial y} = B_z$$



$$\text{But } \text{div} \bar{B} = 0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = -\frac{\partial B_z}{\partial z} \quad \longrightarrow \quad \underbrace{\int_{z_0}^z \frac{\partial B_z}{\partial z} dz + \frac{\partial F}{\partial x} - \frac{\partial G}{\partial y}}_{= B_z(x, y, z) - B_z(x, y, z_0)} = B_z \quad \Rightarrow \quad \frac{\partial F}{\partial x} - \frac{\partial G}{\partial y} = B_z(x, y, z_0)$$

A solution to this equation is:

$$\begin{cases} F(x, y) = 0 \\ G(x, y) = -\int_{y_0}^y B_z(x, y, z_0) dy \end{cases}$$

$$\bar{A}^* = \left(\int_{z_0}^z B_y(x, y, z) dz - \int_{y_0}^y B_z(x, y, z_0) dy, \quad -\int_{z_0}^z B_x(x, y, z) dz, \quad 0 \right)$$

The general solution can be found using $\bar{B} = \text{rot} \bar{A}$:

$$\text{rot}(\bar{A} - \bar{A}^*) = \bar{B} - \bar{B} = 0 \quad \Rightarrow \quad \bar{A} - \bar{A}^* = \text{grad} \psi \quad \Rightarrow \quad \bar{A} = \bar{A}^* + \text{grad} \psi$$

WHICH STATEMENT IS WRONG?

- 1- The curl of a vector field is a scalar (yellow)
- 2- The curl is related to the line integral of a field along a closed surface (red)
- 3- Stokes' theorem translates a line integral into a surface integral (green)
- 4- The Stokes' theorem can be applied only to a closed curve (blue)